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$L(3, 2, 1)$-LABELING OF SOME CYCLE RELATED GRAPHS

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Abstract: Let $G = (V(G), E(G))$ be a connected graph. For $i, j, k \in \mathbb{N}$ with $i \geq j \geq k$, $L(i, j, k)$-labeling of graph G is an integer labeling of the vertices of graph G such that labels of adjacent vertices differ by at least i, labels of vertices at distance two differ by at least j and labels of vertices at distance three differ by at least k. In this paper, we discuss $L(3, 2, 1)$-labeling for crown, arm crown, tadpole, and closed helm graphs.

Keywords and Phrases: Graph Labeling, Cycle Graph, $L(3, 2, 1)$-labeling.

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1. Introduction

The main objective in the set-up of a wireless communication system is the assignment of channels to radio transmitters. A proper channel assignment to radio transmitters that satisfies interference constraints with minimum use of the spectrum is desirable. The inference between two channels is inversely proportional to the distance between transmitters. That is, in a network, if two transmitters are closer than the inference is higher between them. In this case, the channel assigned to these two transmitters must have a large separation to avoid inference. Also, if the distance between two transmitters is sufficiently large, the same channel can be assigned to both transmitters. Hale [9] studied this problem for the first time in 1980, and later it was modified by Roberts [14], which is called the channel assignment problem. Motivated by channel assignment, Griggs and Yeh [8] have defined $L(2,1)$-labeling of a graph. It was defined as follows:

Definition 1.1. *An $L(2,1)$-labeling of a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \mathbb{N} \cup \{0\}$ such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$, and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The span of f is defined as $span f = max\{f(v) | v \in V(G)\}$. Then the $L(2,1)$-labeling number $k(G) = \min_f span(f)$.*

They have also studied the newly defined concept for cycles and trees. Exact λ-number of cycles were also determined, while upper and lower bounds of the radio number of trees in terms of maximum degree were also discussed. They also proved that finding $\lambda(G)$ is an NP-complete problem. Wang [17] gave a sufficient condition for trees to achieve lower bounds for λ-number. Vaidya and Bantva [16] investigated the λ-number for some cacti.

Shao [12] introduced an $L(3,2,1)$-labeling of a graph as a variant of the $L(2,1)$-labeling. It was defined as follows:

Definition 1.2. *An $L(3,2,1)$-labeling of a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \mathbb{N}$ such that $|f(x) - f(y)| \geq 3$ if $d(x,y) = 1$, $|f(x) - f(y)| \geq 2$ if $d(x,y) = 2$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 3$. The span of f is defined as $span f = max\{f(v) | v \in V(G)\}$. Then the $L(3,2,1)$-labeling number $k(G) = \min_f span(f)$.*

Shao [15] and Shao and Liu [12] have given an upper bounds for $L(3,2,1)$-labeling number for various classes of graphs. In 2005, Clipperton *et al.* [6] determined the k-number for the path, cycle, caterpillar, complete graph, and complete bipartite graph. Also, they have given an upper bound for $k(G)$ in terms of the maximum degree of the graph. In [4], Chia *et al.* have studied $L(3,2,1)$-labeling for various graph operations.

Kim *et al.* [10, 11] have investigated $L(3,2,1)$-labeling for the Cartesian product of a complete graph and a cycle, as well as the Cartesian product of a path and a cycle. While, Zhang [8] has discussed $L(3,2,1)$-labeling for various trees. k-number for the fan, double fan, wheel, and friendship graph was investigated by Murugan and Suriya [13]. Amanathulla and Pal [1, 2, 3] have discussed $L(3,2,1)$-labeling for several classes of graph.

In this paper, we obtain an $L(3,2,1)$-labeling number for the crown graph, arm crown graph, tadpole graph, and closed helm graph.

2. Basic Preliminaries

Throughout this paper, we have considered a finite, simple, undirected, and connected graph $G = (V(G), E(G))$. For terminology related to graph theory, we refer to Clark and Holton [5]. For an extensive survey on graph labeling and bibliographic references, we refer Galian [7]. Now, we will list the definitions and results required for the advancement of the paper.

Definition 2.1. *Let G and H be two graphs. The Corona product of G and H, denoted by $G \odot H$, is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i^{th} copies of H to the i^{th} vertex of G, for $i = 1, 2, 3, \ldots, |V(G)|$.*

Definition 2.2. *The crown graph $(C_n \odot K_1)$ is obtained by joining a pendant edge to each vertex of cycle C_n*

Definition 2.3. *The armed crown is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge. It is denoted by ACr_n where n is the number of vertices in cycle C_n.*

Definition 2.4. *The graph obtained by joining cycle C_m to a path P_n by an edge is called tadpole graph. It is denoted by $T(m, n)$.*

Definition 2.5. *The wheel graph W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as the apex vertex and the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges.*

Definition 2.6. *The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 2.7. *The closed helm CH_n is the graph obtained from helm H_n by joining each pendant vertex to form a cycle.*

Proposition 2.8. (Clipperton *et al.* [6]) *For any cycle C_n with $n \geq 3$,*

$$k(C_n) = \begin{cases} 7; & \text{if } n = 3, \\ 8; & \text{if } n \text{ is even}, \\ 9; & \text{if } n \text{ is odd and } n \neq 3, 7, \\ 10; & \text{if } n = 7. \end{cases}$$

Proposition 2.9. (Clipperton *et al.* [6]) *For any complete graph K_n, $k(K_n) = 3n - 2$.*

Proposition 2.10. (Murugan Suriya [13]) *For Wheel graph, W_n*

$$k(W_n) = \begin{cases} 10, & \text{if } n = 3; \\ 11, & \text{if } n = 4; \\ 2(n+1), & \text{if } n \geq 5. \end{cases}$$

3. Main Results

Theorem 3.1. *For crown graph $C_n \odot K_1$ with $n \geq 3$,*

$$k(C_n \odot K_1) = \begin{cases} 9; & \text{if } n \equiv 0 (mod\ 4), \\ 10; & \text{if } n \equiv 2 (mod\ 4), \\ 11; & \text{if } n = 3, \\ 11; & \text{if } n \equiv 1 (mod\ 4) \text{ and } n > 5, \\ 12; & \text{if } n = 5, \\ 12; & \text{if } n \equiv 3 (mod\ 4) \text{ and } n > 5. \end{cases}$$

Proof. Let $C_n \odot K_1$ be the crown graph with vertex set $V(C_n \odot K_1) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ and edge set $E(C_n \odot K_1) = \{v_i v_{i+1}/1 \leq i < n\} \cup \{v_i v'_i/1 \leq i \leq n\} \cup \{v_1 v_n\}$.
We define $f : V(C_n \odot K_1) \to \mathbb{N}$, as per the following nine cases:
Case-1 For $n \equiv 0 (\text{mod } 4)$.
By Proposition 2.8, we have $k(C_n) = 8$, for even n.
Since C_n is a subgraph of $C_n \odot K_1$, $k(C_n \odot K_1) \geq 8$.
It is not possible to label $C_n \odot K_1$ with $\{1, 2, 3, \ldots, 8\}$ and satisfy the condition for $L(3, 2, 1)$-labeling. Thus, $k(C_n \odot K_1) > 8$.
Assign

$$\left.\begin{aligned} f(v_{4i+1}) &= 1 \\ f(v_{4i+2}) &= 6 \\ f(v_{4i+3}) &= 9 \\ f(v_{4i+4}) &= 4 \\ f(v'_{4i+1}) &= 8 \\ f(v'_{4i+2}) &= 3 \\ f(v'_{4i+3}) &= 2 \\ f(v'_{4i+4}) &= 7 \end{aligned}\right\} \text{for } 0 \leq i \leq \frac{n}{4} - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $C_n \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_n \odot K_1) = 9$ for $n \equiv 0 (\text{mod } 4)$.
Case-2 For $n \equiv 2 (\text{mod } 4)$.
By Proposition 2.8, we have $k(C_n) = 8$, for even n.
Since C_n is a subgraph of $C_n \odot K_1$, $k(C_n \odot K_1) \geq 8$. It is not possible to label $C_n \odot K_1$ with $\{1, 2, 3, \ldots, 9\}$ and satisfy the condition for $L(3, 2, 1)$-labeling. Thus, $k(C_n \odot K_1) > 9$. Assign $f(v_1) = 1$, $f(v_2) = 4$, $f(v_3) = 7$, $f(v_4) = 2$, $f(v_5) = 5$,

$f(v_6) = 8$,

$$\left.\begin{aligned} f(v_{4i+3}) &= 1\\ f(v_{4i+4}) &= 6\\ f(v_{4i+5}) &= 3\\ f(v_{4i+6}) &= 8 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1,$$

$f(v'_1) = 6,\ f(v'_2) = 9,\ f(v'_3) = 10, f(v'_4) = 9,\ f(v'_5) = 10,\ f(v'_6) = 3,$

$$\left.\begin{aligned} f(v'_{4i+3}) &= 4\\ f(v'_{4i+4}) &= 9\\ f(v'_{4i+5}) &= 10\\ f(v'_{4i+6}) &= 5 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $C_n \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_n \odot K_1) = 10$ for $n \equiv 2 (\text{mod } 4)$.

Case-3 For $n = 3$.

By Proposition 2.8, we have $k(C_3) = 7$.

Since C_3 is a subgraph of $C_3 \odot K_1$, $k(C_3 \odot K_1) \ge 7$.

For vertices corresponding to cycle, assign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7$.

As $d(v'_1, v_1) = 1$ and $d(v'_1, v_2) = d(v'_1, v_3) = 2$, the minimum possible value for $f(v'_1)$ is 9.

Similarly, the minimum possible values for $f(v'_2)$ is 10 and $f(v'_3)$ is 11.

Now, $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7,\ f(v'_1) = 9, f(v'_2) = 10, f(v'_3) = 11$ is an $L(3,2,1)$-labeling of $C_3 \odot K_1$ with minimum span.

Therefore, $k(C_3 \odot K_1) = 11$.

Case-4 For $n \equiv 1 (\text{mod } 4)$, $n > 5$.

By Proposition 2.8, we have $k(C_n) = 9$, for odd n and $n \ne 3, 7$.

Since C_n is a subgraph of $C_n \odot K_1$, $k(C_n \odot K_1) \ge 9$.

It is not possible to label $C_n \odot K_1$ with $\{1, 2, 3, \ldots, 10\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(C_n \odot K_1) > 10$.

Assign $f(v_1) = 5,\ f(v_2) = 1,\ f(v_3) = 7,\ f(v_4) = 3,\ f(v_5) = 9,$

$$\left.\begin{aligned} f(v_{4i+2}) &= 1\\ f(v_{4i+3}) &= 6\\ f(v_{4i+4}) &= 3\\ f(v_{4i+5}) &= 8 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1,$$

$f(v'_1) = 10,\ f(v'_2) = 9,\ f(v'_3) = 10,\ f(v'_4) = 11,\ f(v'_5) = 5,$

$$\left.\begin{array}{r} f(v'_{4i+2}) = 4 \\ f(v'_{4i+3}) = 11 \\ f(v'_{4i+4}) = 10 \\ f(v'_{4i+5}) = 11 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $C_n \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_n \odot K_1) = 11$ for $n \equiv 1 (\text{mod } 4)$.

Case-5 For $n \equiv 3 (\text{mod } 4)$.

By Proposition 2.8, we have $k(C_n) = 9$, for odd n and $n \neq 3, 7$.

Since C_n is a subgraph of $C_n \odot K_1$, $k(C_n \odot K_1) \geq 9$.

It is not possible to label $C_n \odot K_1$ with $\{1, 2, 3, \dots, 11\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(C_n \odot K_1) > 11$.

Assign

$$\left.\begin{array}{r} f(v_{5i-4}) = 5 \\ f(v_{5i-3}) = 1 \\ f(v_{5i-2}) = 7 \\ f(v_{5i-1}) = 3 \\ f(v_{5i}) = 9 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{r} f(v_{4i+12}) = 1 \\ f(v_{4i+13}) = 6 \\ f(v_{4i+14}) = 3 \\ f(v_{4i+15}) = 8 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4,$$

$$\left.\begin{array}{r} f(v'_{5(i-1)+1}) = 11 \\ f(v'_{5(i-1)+2}) = 10 \\ f(v'_{5(i-1)+3}) = 12 \\ f(v'_{5(i-1)+4}) = 11 \\ f(v'_{5(i-1)+5}) = 12 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{r} f(v'_{4i+12}) = 4 \\ f(v'_{4i+13}) = 10 \\ f(v'_{4i+14}) = 11 \\ f(v'_{4i+15}) = 12 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $C_n \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_n \odot K_1) = 12$ for $n \equiv 3 (\text{mod } 4)$.

Case-6 For $n = 5$.
By Proposition 2.8, we have $k(C_5) = 9$.
Since C_5 is a subgraph of $C_5 \odot K_1$, $k(C_5 \odot K_1) \geq 9$.
For vertices corresponding to cycle, assign $f(v_1) = 5$, $f(v_2) = 1$, $f(v_3) = 7$, $f(v_4) = 3$, $f(v_5) = 9$. (By Proposition 2.8)
As $d(v'_1, v_1) = 1$, $d(v'_1, v_2) = d(v'_1, v_5) = 2$ and $d(v'_1, v_4) = 3$, the minimum possible value for $f(v'_1)$ is 11.
Similarly, the minimum possible values for $f(v'_2)$ is 10, $f(v'_3)$ is 12, $f(v'_4)$ is 11 and $f(v'_5)$ is 12.
Now $f(v_1) = 5, f(v_2) = 1, f(v_3) = 7, f(v_4) = 3, f(v_5) = 9, f(v'_1) = 11, f(v'_2) = 10, f(v'_3) = 12, f(v'_4) = 11, f(v'_5) = 12$ is an $L(3, 2, 1)$-labeling of $C_5 \odot K_1$ with minimum span.
Therefore, $k(C_5 \odot K_1) = 12$.
Case-7 For $n = 7$.
By Proposition 2.8, we have $k(C_7) = 10$.
Since C_7 is a subgraph of $C_7 \odot K_1$, $k(C_7 \odot K_1) \geq 10$.
It is not possible to label $C_7 \odot K_1$ with $\{1, 2, 3, \ldots, 11\}$ and satisfy the condition for $L(3, 2, 1)$-labeling.
Thus, $k(C_7 \odot K_1) > 11$.
Assign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7, f(v_4) = 10, f(v_5) = 3, f(v_6) = 6, f(v_7) = 9, f(v'_1) = 11, f(v'_2) = 12, f(v'_3) = 2, f(v'_4) = 1, f(v'_5) = 8, f(v'_6) = 11, f(v'_7) = 12$.
The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $C_7 \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_7 \odot K_1) = 12$.
Case-8 For $n = 11$.
By Proposition 2.8, we have $k(C_{11}) = 9$.
Since C_{11} is a subgraph of $C_{11} \odot K_1$, $k(C_{11} \odot K_1) \geq 9$.
It is not possible to label $C_{11} \odot K_1$ with $\{1, 2, 3, \ldots, 11\}$ and satisfy the condition for $L(3, 2, 1)$-labeling.
Thus, $k(C_{11} \odot K_1) > 11$.
Assign $f(v_1) = 1, f(v_2) = 6, f(v_3) = 3, f(v_4) = 8, f(v_5) = 5, f(v_6) = 1, f(v_7) = 9, f(v_8) = 6, f(v_9) = 2, f(v_{10}) = 8, f(v_{11}) = 4, f(v'_1) = 9, f(v'_2) = 10, f(v'_3) = 11, f(v'_4) = 12, f(v'_5) = 10, f(v'_6) = 7, f(v'_7) = 3, f(v'_8) = 10, f(v'_9) = 11, f(v'_{10}) = 12, f(v'_{11}) = 11$.
The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $C_{11} \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_{11} \odot K_1) = 12$.
Case-9 For $n = 15$.

By Proposition 2.8, we have $k(C_{15}) = 9$.
Since C_{15} is a subgraph of $C_{15} \odot K_1$, $k(C_{15} \odot K_1) \geq 9$.
It is not possible to label $C_{15} \odot K_1$ with $\{1, 2, 3, \ldots, 11\}$ and satisfy the condition for $L(3, 2, 1)$-labeling.
Thus, $k(C_{15} \odot K_1) > 11$.
Assign

$$\left.\begin{array}{l} f(v_{5(i-1)+1}) = 5 \\ f(v_{5(i-1)+2}) = 1 \\ f(v_{5(i-1)+3}) = 7 \\ f(v_{5(i-1)+4}) = 3 \\ f(v_{5(i-1)+5}) = 9 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{l} f(v'_{5(i-1)+1}) = 11 \\ f(v'_{5(i-1)+2}) = 10 \\ f(v'_{5(i-1)+3}) = 12 \\ f(v'_{5(i-1)+4}) = 11 \\ f(v'_{5(i-1)+5}) = 12 \end{array}\right\} \text{for } 1 \leq i \leq 3.$$

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $C_{15} \odot K_1$. In addition, it uses minimum labels. Thus, $k(C_{15} \odot K_1) = 12$. Hence, the result.

Theorem 3.2. *For arm crown graph ACr_n with $n \geq 3$,*

$$k(ACr_n) = \begin{cases} 9; & \text{if } n \equiv 0 (\text{mod } 4), \\ 10; & \text{if } n \equiv 2 (\text{mod } 4), \\ 11; & \text{if } n = 3, \\ 11; & \text{if } n \equiv 1 (\text{mod } 4) \text{ and } n > 5, \\ 12; & \text{if } n = 5, \\ 12; & \text{if } n \equiv 3 (\text{mod } 4) \text{ and } n > 5. \end{cases}$$

Proof. Let ACr_n be the arm crown graph with vertex set $V(ACr_n) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, v''_1, v''_2, \ldots, v''_n\}$ and edge set $E(ACr_n) = \{v_i v_{i+1}/1 \leq i < n\} \cup \{v_1 v_n\} \cup \{v_i v'_i/1 \leq i \leq n\} \cup \{v'_i v''_i/1 \leq i \leq n\}$.
We define $f : V(ACr_n) \to \mathbb{N}$, as per the following nine cases:
<u>**Case-1**</u> For $n \equiv 0(\text{mod } 4)$.
Since $C_n \odot K_1$ is a subgraph of ACr_n and according to Theorem 2.1, $k(C_n \odot K_1) = 9$ for $n \equiv 0(\text{mod } 4)$.

Therefore, $k(ACr_n) \geq 9$.

Assign

$$\left.\begin{array}{l} f(v_{4i+1}) = 1 \\ f(v_{4i+2}) = 6 \\ f(v_{4i+3}) = 9 \\ f(v_{4i+4}) = 4 \end{array}\right\} \text{for } 0 \leq i \leq \frac{n}{4} - 1,$$

$$\left.\begin{array}{l} f(v'_{4i+1}) = 8 \\ f(v'_{4i+2}) = 3 \\ f(v'_{4i+3}) = 2 \\ f(v'_{4i+4}) = 7 \end{array}\right\} \text{for } 0 \leq i \leq \frac{n}{4} - 1,$$

$$\left.\begin{array}{l} f(v''_{4i+1}) = 3 \\ f(v''_{4i+2}) = 8 \\ f(v''_{4i+3}) = 5 \\ f(v''_{4i+4}) = 2 \end{array}\right\} \text{for } 0 \leq i \leq \frac{n}{4} - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of ACr_n. In addition, it uses minimum labels. Thus, $k(ACr_n) = 9$ for $n \equiv 0(\text{mod } 4)$.

Case-2 For $n \equiv 2(\text{mod } 4)$.

Since $C_n \odot K_1$ is a subgraph of ACr_n and according to Theorem 2.1, $k(C_n \odot K_1) = 10$ for $n \equiv 2(\text{mod } 4)$.

Therefore, $k(ACr_n) \geq 10$.

Assign $f(v_1) = 1,\ f(v_2) = 4,\ f(v_3) = 7,\ f(v_4) = 2,\ f(v_5) = 5,\ f(v_6) = 8,$

$$\left.\begin{array}{l} f(v_{4i+3}) = 1 \\ f(v_{4i+4}) = 6 \\ f(v_{4i+5}) = 3 \\ f(v_{4i+6}) = 8 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 1,$$

$$f(v'_1) = 6,\ f(v'_2) = 9,\ f(v'_3) = 10, f(v'_4) = 9,\ f(v'_5) = 10,\ f(v'_6) = 3,$$

$$\left.\begin{array}{r} f(v'_{4i+3}) = 4 \\ f(v'_{4i+4}) = 9 \\ f(v'_{4i+5}) = 10 \\ f(v'_{4i+6}) = 5 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 1,$$

$$f(v''_1) = 3,\ f(v''_2) = 2,\ f(v''_3) = 1, f(v''_4) = 4,\ f(v''_5) = 1,\ f(v''_6) = 6,$$

$$\left.\begin{aligned} f(v''_{4i+3}) &= 7 \\ f(v''_{4i+4}) &= 2 \\ f(v''_{4i+5}) &= 1 \\ f(v''_{4i+6}) &= 2 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of ACr_n. In addition, it uses minimum labels. Thus, $k(ACr_n) = 10$ for $n \equiv 2 (\text{mod } 4)$.

Case-3 For $n = 3$.

Since $C_3 \odot K_1$ is a subgraph of ACr_3 and according to Theorem 2.1, $k(C_3 \odot K_1) = 11$. Therefore, $k(ACr_3) \ge 11$.

Now, $f\colon V(ACr_3) \to \{1, 2, \ldots, 11\}$ defined by $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7$, $f(v'_1) = 9, f(v'_2) = 10, f(v'_3) = 11$, $f(v''_1) = 3, f(v''_2) = 2$ and $f(v''_3) = 2$.

Therefore, $k(ACr_3) = 11$.

Case-4 For $n \equiv 1 (\text{mod } 4)$ and $n > 5$.

Since $C_n \odot K_1$ is a subgraph of ACr_n and according to Theorem 2.1, $k(C_n \odot K_1) = 11$ for $n \equiv 1 (\text{mod } 4)$ and $n > 5$.

Therefore, $k(ACr_n) \ge 11$.

Assign $f(v_1) = 5$, $f(v_2) = 1$, $f(v_3) = 7$, $f(v_4) = 3$, $f(v_5) = 9$,

$$\left.\begin{aligned} f(v_{4i+2}) &= 1 \\ f(v_{4i+3}) &= 6 \\ f(v_{4i+4}) &= 3 \\ f(v_{4i+5}) &= 8 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1,$$

$f(v'_1) = 10$, $f(v'_2) = 9$, $f(v'_3) = 10$, $f(v'_4) = 11$, $f(v'_5) = 5$,

$$\left.\begin{aligned} f(v'_{4i+2}) &= 4 \\ f(v'_{4i+3}) &= 11 \\ f(v'_{4i+4}) &= 10 \\ f(v'_{4i+5}) &= 11 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1,$$

$f(v''_1) = 2$, $f(v''_2) = 3$, $f(v''_3) = 4$, $f(v''_4) = 1$, $f(v''_5) = 2$,

$$\left.\begin{aligned} f(v''_{4i+2}) &= 7 \\ f(v''_{4i+3}) &= 2 \\ f(v''_{4i+4}) &= 1 \\ f(v''_{4i+5}) &= 2 \end{aligned}\right\} \text{for } 1 \le i \le \left\lceil \frac{n}{4} \right\rceil - 1.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of ACr_n. In addition, it uses minimum labels. Thus, $k(ACr_n) = 11$ for $n \equiv 1 (\text{mod } 4)$.

Case-5 For $n \equiv 3 (\text{mod } 4)$ and $n > 5$.

Since $C_n \odot K_1$ is a subgraph of ACr_n and according to Theorem 2.1, $k(C_n \odot K_1) = 12$ for $n \equiv 3 (\text{mod } 4)$ and $n > 5$.
Assign

$$\left.\begin{array}{l} f(v_{5(i-1)+1}) = 5 \\ f(v_{5(i-1)+2}) = 1 \\ f(v_{5(i-1)+3}) = 7 \\ f(v_{5(i-1)+4}) = 3 \\ f(v_{5(i-1)+5}) = 9 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{l} f(v_{4i+12}) = 1 \\ f(v_{4i+13}) = 6 \\ f(v_{4i+14}) = 3 \\ f(v_{4i+15}) = 8 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4,$$

$$\left.\begin{array}{l} f(v'_{5(i-1)+1}) = 11 \\ f(v'_{5(i-1)+2}) = 10 \\ f(v'_{5(i-1)+3}) = 12 \\ f(v'_{5(i-1)+4}) = 11 \\ f(v'_{5(i-1)+5}) = 12 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{l} f(v'_{4i+12}) = 4 \\ f(v'_{4i+13}) = 10 \\ f(v'_{4i+14}) = 11 \\ f(v'_{4i+15}) = 12 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4,$$

$$\left.\begin{array}{l} f(v''_{5(i-1)+1}) = 2 \\ f(v''_{5(i-1)+2}) = 3 \\ f(v''_{5(i-1)+3}) = 2 \\ f(v''_{5(i-1)+4}) = 5 \\ f(v''_{5(i-1)+5}) = 2 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{l} f(v''_{4i+12}) = 7 \\ f(v''_{4i+13}) = 2 \\ f(v''_{4i+14}) = 5 \\ f(v''_{4i+15}) = 2 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4.$$

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of ACr_n. In addition, it uses minimum

labels. Thus, $k(ACr_n) = 12$ for $n \equiv 3(\text{mod } 4)$.
Case-6 For $n = 5$.
Since $C_5 \odot K_1$ is a subgraph of ACr_5 and according to Theorem 2.1, $k(C_5 \odot K_1) = 12$.
Assign $f(v_1) = 5, f(v_2) = 1, f(v_3) = 7, f(v_4) = 3, f(v_5) = 9, f(v'_1) = 11, f(v'_2) = 10, f(v'_3) = 12, f(v'_4) = 11, f(v'_5) = 12, f(v''_1) = 2, f(v''_2) = 3, f(v''_3) = 4, f(v''_4) = 1, f(v''_5) = 1$.
The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of ACr_5. In addition, it uses minimum labels. Thus, $k(ACr_5) = 12$.
Case-7 For $n = 7$.
Since $C_7 \odot K_1$ is a subgraph of ACr_7 and according to Theorem 2.1, $k(C_7 \odot K_1) = 12$.
Assign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7, f(v_4) = 10, f(v_5) = 3, f(v_6) = 6, f(v_7) = 9, f(v'_1) = 11, f(v'_2) = 12, f(v'_3) = 2, f(v'_4) = 1, f(v'_5) = 8, f(v'_6) = 11, f(v'_7) = 12, f(v''_1) = 3, f(v''_2) = 2, f(v''_3) = 5, f(v''_4) = 4, f(v''_5) = 1, f(v''_6) = 1, f(v''_7) = 2$.
The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of ACr_7. In addition, it uses minimum labels. Thus, $k(ACr_7) = 12$.
Case-8 For $n = 11$.
Since $C_{11} \odot K_1$ is a subgraph of ACr_{11} and according to Theorem 2.1, $k(C_{11} \odot K_1) = 12$.
Assign $f(v_1) = 1, f(v_2) = 6, f(v_3) = 3, f(v_4) = 8, f(v_5) = 5, f(v_6) = 1, f(v_7) = 9, f(v_8) = 6, f(v_9) = 2, f(v_{10}) = 8, f(v_{11}) = 4, f(v'_1) = 9, f(v'_2) = 10, f(v'_3) = 11, f(v'_4) = 12, f(v'_5) = 10, f(v'_6) = 7, f(v'_7) = 3, f(v'_8) = 10, f(v'_9) = 11, f(v'_{10}) = 12, f(v'_{11}) = 11, f(v''_1) = 3, f(v''_2) = 2, f(v''_3) = 1, f(v''_4) = 1, f(v''_5) = 2, f(v''_6) = 3, f(v''_7) = 7, f(v''_8) = 1, f(v''_9) = 4, f(v''_{10}) = 1, f(v''_{11}) = 2$.
The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of ACr_{11}. In addition, it uses minimum labels. Thus, $k(ACr_{11}) = 12$.
Case-9 For $n = 15$.
Since $C_{15} \odot K_1$ is a subgraph of ACr_{15} and according to Theorem 2.1, $k(C_{15} \odot K_1) = 12$.
Assign

$$\left.\begin{aligned} f(v_{5(i-1)+1}) &= 5 \\ f(v_{5(i-1)+2}) &= 1 \\ f(v_{5(i-1)+3}) &= 7 \\ f(v_{5(i-1)+4}) &= 3 \\ f(v_{5(i-1)+5}) &= 9 \end{aligned}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{aligned} f(v'_{5(i-1)+1}) &= 11\\ f(v'_{5(i-1)+2}) &= 10\\ f(v'_{5(i-1)+3}) &= 12\\ f(v'_{5(i-1)+4}) &= 11\\ f(v'_{5(i-1)+5}) &= 12 \end{aligned}\right\} \text{for } 1 \le i \le 3,$$

$$\left.\begin{aligned} f(v''_{5(i-1)+1}) &= 2\\ f(v''_{5(i-1)+2}) &= 3\\ f(v''_{5(i-1)+3}) &= 2\\ f(v''_{5(i-1)+4}) &= 1\\ f(v''_{5(i-1)+5}) &= 1 \end{aligned}\right\} \text{for } 1 \le i \le 3.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of ACr_{15}. In addition, it uses minimum labels. Thus, $k(ACr_{15}) = 12$. Hence, the result.

Theorem 3.3. *For tadpole graph $T(m,n)$ with $m \ge 3$ and $n \ge 1$,*

$$k(T(m,n)) = \begin{cases} 8; & \text{if } m = 3 \text{ or } m \text{ is even,}\\ 9; & \text{if } m \text{ is odd and } m \ge 9,\\ 10; & \text{if } m = 5, 7. \end{cases}$$

Proof. Let $T(m,n)$ be the tadpole graph with vertex set $V(T(m,n)) = \{v_i/i = 1,2,\ldots,m\} \cup \{u_j/j = 1,2,\ldots,n\}$ and edge set $E(T(m,n)) = \{v_iv_{i+1}/i = 1,2,\ldots, m-1\} \cup \{v_1v_m, v_1u_1\} \cup \{u_iu_{i+1}/i = 1,2,\ldots,n-1\}$.
We define $f : V(T(m,n)) \to \mathbb{N}$, as per the following five cases:
Case-1 For $m = 3$.
By Proposition 2.8, we have $k(C_3) = 7$.
Since C_3 is a subgraph of $T(m,n)$, $k(T(m,n)) \ge 7$.
If we assign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7$. Then we must assign $f(u_1) = 9$. But we want a minimum label to assign.
Therefore, we reassign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 8$, $f(u_1) = 6, f(u_2) = 3, f(u_3) = 8, f(u_4) = 1$,

$$f(u_{4i+1}) = 6 \text{ for } 1 \le i \le \left\lfloor \frac{n-1}{4} \right\rfloor,$$

$$f(u_{4i+2}) = 3 \text{ for } 1 \le i \le \left\lfloor \frac{n-2}{4} \right\rfloor,$$

$$f(u_{4i+3}) = 8 \text{ for } 1 \le i \le \left\lfloor \frac{n-3}{4} \right\rfloor,$$

$f(u_{4i+4}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $T(m,n)$. In addition, it uses minimum labels. Thus, $k(T(m,n)) = 8$ for $m = 3$.

Case-2 For even m.

Subcase-I For $m \equiv 0 (\text{mod } 4)$ with $m \geq 4$.

By Proposition 2.8, we have $k(C_m) = 8$, for even m.

Since C_m is a subgraph of $T(m,n)$, $k(T(m,n)) \geq 8$.

Assign

$$\left.\begin{array}{l} f(v_{4i+1}) = 1 \\ f(v_{4i+2}) = 6 \\ f(v_{4i+3}) = 3 \\ f(v_{4i+4}) = 8 \end{array}\right\} \text{for } 0 \leq i \leq \frac{m}{4} - 1,$$

$f(u_1) = 4, f(u_2) = 7, f(u_3) = 2, f(u_4) = 5, f(u_5) = 8$,

$f(u_{4i+2}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor$,

$f(u_{4i+3}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor$,

$f(u_{4i+4}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor$,

$f(u_{4i+5}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n-5}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $T(m,n)$. In addition, it uses minimum labels. Thus, $k(T(m,n)) = 8$ for $m \equiv 0 (\text{mod } 4)$ with $m \geq 4$.

Subcase-II For $m \equiv 2 (\text{mod } 4)$ with $m \geq 10$.

By Proposition 2.8, we have $k(C_m) = 8$, for even m.

Since C_m is a subgraph of $T(m,n)$, $k(T(m,n)) \geq 8$

Assign $f(v_1) = 1, f(v_2) = 4, f(v_3) = 7, f(v_4) = 2, f(v_5) = 5, f(v_6) = 8$,

$$\left.\begin{array}{l} f(v_{4i+3}) = 1 \\ f(v_{4i+4}) = 6 \\ f(v_{4i+5}) = 3 \\ f(v_{4i+6}) = 8 \end{array}\right\} \text{for } 1 \leq i \leq \frac{m-6}{4},$$

$f(u_1) = 6, f(u_2) = 3, f(u_3) = 8, f(u_4) = 1$,

$f(u_{4i+1}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$,

$f(u_{4i+2}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor$,

$f(u_{4i+3}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor$,

$f(u_{4i+4}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $T(m, n)$. In addition, it uses minimum labels. Thus, $k(T(m, n)) = 8$ for $m \equiv 2 \pmod 4$ with $m \geq 10$.

Case-3 For odd $m \geq 9$.

Subcase-I For $m \equiv 1 \pmod 4$ and $m \geq 9$.

By Proposition 2.8, we have $k(C_m) = 9$, for odd m.

Since C_m is a subgraph of $T(m, n)$, $k(T(m, n)) \geq 9$.

Assign $f(v_1) = 5, f(v_2) = 1, f(v_3) = 7, f(v_4) = 3, f(v_5) = 9$,

$$\left.\begin{aligned} f(v_{4i+2}) &= 1 \\ f(v_{4i+3}) &= 6 \\ f(v_{4i+4}) &= 3 \\ f(v_{4i+5}) &= 8 \end{aligned}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{m}{4} \right\rceil - 1,$$

$f(u_1) = 9, f(u_2) = 6, f(u_3) = 3, f(u_4) = 8$,

$f(u_{4i+1}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$,

$f(u_{4i+2}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor$,

$f(u_{4i+3}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor$,

$f(u_{4i+4}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $T(m, n)$. In addition, it uses minimum labels. Thus, $k(T(m, n)) = 9$ for $m \equiv 1 \pmod 4$ with $m \geq 9$.

Subcase-II For $m \equiv 3 \pmod 4$ with $m \geq 19$.

By Proposition 2.8, we have $k(C_m) = 9$, for odd m.

Since C_m is a subgraph of $T(m, n)$, $k(T(m, n)) \geq 9$.

Assign

$$\left.\begin{aligned} f(v_{5i-4}) &= 5 \\ f(v_{5i-3}) &= 1 \\ f(v_{5i-2}) &= 7 \\ f(v_{5i-1}) &= 3 \\ f(v_{5i}) &= 9 \end{aligned}\right\} \text{for } 1 \leq i \leq 3,$$

$$\left.\begin{array}{l} f(v_{4i+12}) = 1 \\ f(v_{4i+13}) = 6 \\ f(v_{4i+14}) = 3 \\ f(v_{4i+15}) = 8 \end{array}\right\} \text{for } 1 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 4,$$

$f(u_1) = 9, f(u_2) = 6, f(u_3) = 3, f(u_4) = 8,$

$f(u_{4i+1}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$,

$f(u_{4i+2}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n-2}{4} \right\rfloor$,

$f(u_{4i+3}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n-3}{4} \right\rfloor$,

$f(u_{4i+4}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n-4}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $T(m, n)$. In addition, it uses minimum labels. Thus, $k(T(m, n)) = 9$ for $m \equiv 3 (\text{mod } 4)$ with $m \geq 19$.

Subcase-III For $m = 11$.

By Proposition 2.8, we have $k(C_{11}) = 9$.

Since C_{11} is a subgraph of $T(11, n)$, $k(T(11, n)) \geq 9$.

Assign $f(v_1) = 1, f(v_2) = 6, f(v_3) = 3, f(v_4) = 8, f(v_5) = 5, f(v_6) = 1, f(v_7) = 9, f(v_8) = 6, f(v_9) = 2, f(v_{10}) = 8, f(v_{11}) = 4, f(u_1) = 9,$

$f(u_{4i-2}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$,

$f(u_{4i-1}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n+1}{4} \right\rfloor$,

$f(u_{4i}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$,

$f(u_{4i+1}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3, 2, 1)$-labeling. Thus, f is an $L(3, 2, 1)$-labeling of $T(11, n)$. In addition, it uses minimum labels. Thus, $k(T(11, n)) = 9$.

Subcase-IV For $m = 15$.

By Proposition 2.8, we have $k(C_{15}) = 9$.

Since C_{15} is a subgraph of $C_{15} \odot K_1$, $k(T(15, n)) \geq 9$.

Assign

$$\left.\begin{array}{l} f(v_{5(i-1)+1}) = 1 \\ f(v_{5(i-1)+2}) = 7 \\ f(v_{5(i-1)+3}) = 3 \\ f(v_{5(i-1)+4}) = 9 \\ f(v_{5(i-1)+5}) = 5 \end{array}\right\} \text{for } 1 \leq i \leq 3,$$

$f(u_1) = 9$,

$f(u_{4i-2}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$,

$f(u_{4i-1}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n+1}{4} \right\rfloor$,

$f(u_{4i}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$,

$f(u_{4i+1}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $T(15,n)$. In addition, it uses minimum labels. Thus, $k(T(15,n)) = 9$.

Case-4 For $m = 5$.

By Proposition 2.8, we have $k(C_5) = 9$.

Since C_5 is a subgraph of $T(5,n)$, $k(T(5,n)) \geq 9$.

For vertices corresponding to cycle, assign $f(v_1) = 1$, $f(v_2) = 7$, $f(v_3) = 3$, $f(v_4) = 9$, $f(v_5) = 5$. (By Proposition 2.8) Then in all possible labeling, the minimum label we can assign to $f(u_1)$ is 10. Thus,

$f(u_1) = 10$,

$f(u_{4i-2}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$,

$f(u_{4i-1}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n+1}{4} \right\rfloor$,

$f(u_{4i}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$,

$f(u_{4i+1}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n-1}{4} \right\rfloor$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $T(5,n)$. In addition, it uses minimum labels. Thus, $k(T(5,n)) = 10$.

Case-5 For $m = 7$.

By Proposition 2.8, we have $k(C_7) = 10$.

Since C_7 is a subgraph of $T(7,n)$, $k(T(7,n)) \geq 10$.

Assign $f(v_1) = 1, f(v_2) = 8, f(v_3) = 5, f(v_4) = 2, f(v_5) = 10, f(v_6) = 7, f(v_7) = 4$,

$f(u_{4i-3}) = 6$ for $1 \leq i \leq \left\lfloor \frac{n+3}{4} \right\rfloor$,
$f(u_{4i-2}) = 3$ for $1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$,
$f(u_{4i-1}) = 8$ for $1 \leq i \leq \left\lfloor \frac{n+1}{4} \right\rfloor$,
$f(u_{4i}) = 1$ for $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$.
The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of $T(7,n)$. In addition, it uses minimum labels. Thus, $k(T(7,n)) = 10$. Hence, the result.

Theorem 3.4. *For closed helm graph CH_n with $n \geq 3$,*

$$k(CH_n) = \begin{cases} 13; & \text{if } n = 3 \text{ and } 4\ , \\ 15; & \text{if } n = 6\ , \\ 16; & \text{if } n = 5\ , \\ 2(n+1); & \text{if } n \geq 7. \end{cases}$$

Proof. Let CH_n be the closed helm graph with vertex set $V(CH_n) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n, v\}$ and edge set $E(CH_n) = \{v_i v_{i+1}/1 \leq i < n\} \cup \{v'_i v'_{i+1}/1 \leq i < n\} \cup \{v_i v'_i/1 \leq i \leq n\} \cup \{v v_i/1 \leq i \leq n\} \cup \{v_1 v_n\} \cup \{v'_1 v'_n\}$.
We define $f : V(CH_n) \to \mathbb{N}$, as per the following five cases:
Case-1 For $n = 3$.
Since W_3 is a subgraph of CH_3 and by Proposition 2.10, $k(W_3) = 10$. Therefore, $k(CH_3) \geq 10$.
It is not possible to label CH_3 with $\{1, 2, 3, \ldots, 12\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(CH_3) > 12$.
Assign $f(v_1) = 5$, $f(v_2) = 9$, $f(v_3) = 13$, $f(v) = 1$, $f(v'_1) = 11$, $f(v'_2) = 3$ and $f(v'_3) = 7$.
The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of CH_3. In addition, it uses minimum labels. Thus, $k(CH_3) = 13$.
Case-2 For $n = 4$.
Since W_4 is a subgraph of CH_4 and by Proposition 2.10, $k(W_4) = 11$. Therefore, $k(CH_4) \geq 11$.
It is not possible to label CH_4 with $\{1, 2, 3, \ldots, 12\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(CH_4) > 12$.
Assign $f(v_1) = 9$, $f(v_2) = 4$, $f(v_3) = 7$, $f(v_4) = 12$, $f(v) = 1$, $f(v'_1) = 6$, $f(v'_2) = 13$, $f(v'_3) = 10$ and $f(v'_4) = 3$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of CH_4. In addition, it uses minimum labels. Thus, $k(CH_4) = 13$.

Case-3 For $n = 6$.

Since W_6 is a subgraph of CH_6 and by Proposition 2.10, $k(W_6) = 14$. Therefore, $k(CH_6) \geq 14$.

It is not possible to label CH_6 with $\{1, 2, 3, \ldots, 14\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(CH_6) > 14$.

Assign $f(v_1) = 1$, $f(v_2) = 10$, $f(v_3) = 5$, $f(v_4) = 12$, $f(v_5) = 3$, $f(v_6) = 8$, $f(v) = 15$, $f(v_1') = 4$, $f(v_2') = 7$, $f(v_3') = 2$, $f(v_4') = 9$, $f(v_5') = 6$ and $f(v_6') = 11$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of CH_6. In addition, it uses minimum labels. Thus, $k(CH_6) = 15$.

Case-4 For $n = 5$.

Since W_5 is a subgraph of CH_5 and $k(W_5) = 12$. Therefore, $k(CH_5) \geq 12$.

It is not possible to label CH_6 with $\{1, 2, 3, \ldots, 15\}$ and satisfy the condition for $L(3,2,1)$-labeling. Thus, $k(CH_5) > 15$.

Assign $f(v_1) = 1$, $f(v_2) = 4$, $f(v_3) = 7$, $f(v_4) = 10$, $f(v_5) = 13$, $f(v) = 16$, $f(v_1') = 8$, $f(v_2') = 11$, $f(v_3') = 14$, $f(v_4') = 2$ and $f(v_5') = 5$.

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of CH_5. In addition, it uses minimum labels. Thus, $k(CH_5) = 16$.

Case-5 For $n \geqslant 7$.

Since W_n is a subgraph of CH_n and $k(W_n) = 2(n+1)$ for $n \geqslant 7$. Therefore, $k(CH_n) \geq 2(n+1)$.

Define $f \colon V(CH_n) \to \{1, 2, ..., 2(n+1)\}$ as

Subcase-I For even n.

$$\begin{aligned}
f(v_{2i-1}) &= 2i - 1; && \text{for } 1 \leq i \leq \frac{n}{2},\\
f(v_{2i}) &= 2i + (n-1); && \text{for } 1 \leq i \leq \frac{n}{2},\\
f(v) &= 2(n+1), &&\\
f(v'_{2i-1}) &= 2i + 2; && \text{for } 1 \leq i \leq \frac{n-2}{2},\\
f(v'_{2i}) &= 2i + (n+2); && \text{for } 1 \leq i \leq \frac{n-2}{2},\\
f(v'_{n-1}) &= 2, &&\\
f(v'_n) &= n + 2. &&
\end{aligned}$$

Subcase-II For odd n.

$$f(v_{2i-1}) = 2i - 1; \quad \text{for } 1 \leq i \leq \frac{n+1}{2},$$
$$f(v_{2i}) = 2i + n; \quad \text{for } 1 \leq i \leq \frac{n-1}{2},$$
$$f(v) = 2(n+1),$$
$$f(v'_{2i-1}) = 2i + 2; \quad \text{for } 1 \leq i \leq \frac{n+1}{2},$$
$$f(v'_{2i}) = 2i + (n+3); \quad \text{for } 1 \leq i \leq \frac{n-3}{2},$$
$$f(v'_{n-1}) = 2.$$

The above defined vertex labeling function satisfies the condition for $L(3,2,1)$-labeling. Thus, f is an $L(3,2,1)$-labeling of CH_n. In addition, it uses minimum labels. Thus, $k(CH_n) = 2(n+1)$. Hence, the result.

4. Concluding Remarks

The $L(3,2,1)$-number k is completely determined for four new graph families. This work is an effort to relate the $L(3,2,1)$-number of cycle, and the larger graphs obtained from cycle. Thus, gives a better idea for the expansion of wireless networks without inferences.

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